A Stochastic Filter for Planar Rigid-Body Motions

Igor Gilitschenski*, Gerhard Kurz*, and Uwe D. Hanebeck*

Abstract— **This paper presents a novel algorithm for the estimation of planar rigid-body motions. It is based on using a probability distribution that is inherently defined on the nonlinear manifold representing these motions and on proposing a deterministic sampling scheme that makes consideration of complicated system models possible. Furthermore, we show that the measurement update for a manifold equivalent to noisy direct measurements can be carried out in closed form. Thus, the resulting method avoids errors made due to local linearization and outperforms methods that wrongly assume Gaussian distributions, which we show by comparing the proposed filter to the UKF.**

I. Introduction

Estimation of rigid-body motions is an inherently nonlinear filtering problem that occurs in a wide area of applications such as robotics, intelligent mobility, or mixed and augmented reality. The underlying nonlinearity is not necessarily due to complicated system dynamics but a mere consequence of the fact that the domain of rigid-body motions is nonlinear itself. Thus, a sound consideration of this underlying domain promises to yield robust and intuitively suitable dynamic state estimation algorithms.

Most approaches that are based on some nonlinear variant of the Kalman filter usually make use of local linearity of the underlying manifold. This is justified whenever all arising noise is sufficiently low, and thus, the underlying state space can be approximated by a linear space. In that situation, it is may be well justified to make use of approaches assuming Gaussian uncertainty. In order to ensure the resulting estimate to be valid, projection techniques are usually employed such as those discussed in [1]. These approaches may perform well in some practical applications. However, their inherent problem is the lack of a probabilistic interpretation. Sound probabilistic consideration of nonlinear domains is made possible by making use of directional statistics [2], [3], which is a subfield of statistics that considers quantities and probability distributions defined on nonlinear manifolds such as circles, spheres or—as considered in this work—rigid-body motions.

The filter presented in this work is based on [4]. It uses unit dual quaternions for representation of rigid-body motions. Dual quaternions are a natural extension of quaternions, which was proposed by Clifford in [5]. This choice makes it possible to encode position and orientation into four values in the planar case (or eight values in the three-dimensional case). This representation is used within a probability distribution for representing uncertain rigid-body motions, which was proposed in [6].

Building upon this theory, this paper has two main contributions. First, a deterministic sampling scheme for the considered distribution is proposed, i.e., for deterministically approximating the continuous distribution by a discrete distribution that is defined on the same domain. It is shown how deterministic sampling of the proposed distribution can be reduced to deterministically sampling a Gaussian and a Bingham distribution [7], which is an antipodally symmetric distribution on the unit hypersphere. Second, the resulting sampling scheme is used for developing a sample-based filter that is reminiscent of Gaussian sigma-point filters, such as the unscented Kalman filter (UKF) [8]. The resulting filter is not only capable of considering non-identity system models but also makes a closed-form measurement update possible.

Due to the fact that local linearization is well justified in case of sufficiently low noise, there has been also a number of works about filters based on this assumption for performing estimation of quantities defined on nonlinear domains. For example, in the case of orientation, quaternion-based filters have been discussed in [9], [10], [11], [12], [13]. Furthermore, pose estimation based on dual-quaternions has been investigated in [14]. One of the earliest approaches that uses directional statistics for dynamic state estimation was proposed in [15]. However, most of these approaches have been developed in the recent years. This involves algorithms for estimating circular quantities [16], [17] or orientations [18], [7]. For consideration of rigid-body motions, there are two particularly notable works. In [19], projected Gaussians are used for uncertainty representation. This approach has also the advantage of using dual-quaternions, and thus, offering a compact representation. However, it suffers from the fact that the Bayes update is not possible in closed-form. In [20], a probability distribution was proposed that is similar to the distribution used in this work. However, this work does not involve a deterministic sampling scheme and requires the use of highly redundant rotation matrices for orientation representation.

The remainder of this paper is structured as follows. In Sec. II we provide an introduction to representing rigid-body motions with dual quaternions and we also revisit the probability distribution that will be used for representing uncertainties for this kind of motions. Sec. III provides a hybrid deterministic sampling scheme for this distribution

*I. Gilitschenski is with Autonomous Systems Laboratory (ASL), Institute of Robotics and Intelligent Systems, Swiss Federal Institute of Technology (ETH) Zurich, 8092 Zurich, Switzerland (e-mail: igilitschenski@ethz.ch). G. Kurz and U. D. Hanebeck are with Intelligent Sensor-Actuator-Systems Laboratory (ISAS), Institute of Anthropomatics and Robotics, Karlsruhe Institute of Technology (KIT), 76131 Karlsruhe, Germany (e-mail: gerhard.kurz@kit.edu; uwe.hanebeck@ieee.org).

that can be used for approximate numerical integration when computing expectation values. All of this is brought together in Sec. IV for developing the new filter that is based on a prediction and update step. The filter is evaluated and compared against the UKF in Sec. V. The presented results are discussed and concluded in Sec. VI.

II. Uncertain Rigid-Body Motions

For deriving of our filter, we revisit two preliminary topics. First, we explain the concept of dual quaternions and their applicability for the representation of rigid-body motions. The group that is used for representation of these motions is known as the *Special Euclidean Group* (it is abbreviated as SE(3) for the $3d$ case and SE(2) for the $2d$ case). A multiplicative subgroup of the dual quaternions is derived that can be thought of as a double cover of SE(2). It is assumed that the reader is familiar with quaternions and their applicability to representing rotations. A good introduction can be found in [21]. Second, a probability distribution is revisited that can be used for the representation of this kind of motions. It is based on partially conditioning a $4d$ Gaussian distribution. Thus, it is related to both, the Gaussian distribution and the Bingham distribution [22], which arises by conditioning a zero-mean Gaussian vector to unit length.

A. Representation of Planar Rigid-Body Motions

Dual quaternions combine the concept of quaternions and dual numbers. The latter are an extension of the real numbers that introduces a dual unit ε which is characterized by the nilpotency property $\varepsilon^2 = 0$. In combination with quaternions, the dual unit ε commutes with quaternion units i, j, k, e.g., $\varepsilon \mathrm{i} = \mathrm{i}\varepsilon$. The resulting dual quaternion can be described by the entries of a vector $\underline{a} \in \mathbb{R}^8$, i.e.,

$$\begin{aligned} &a^{(1)} + a^{(2)}\mathrm{i} + a^{(3)}\mathrm{j} + a^{(4)}\mathrm{k} \\ &\quad + \varepsilon(a^{(5)} + a^{(6)}\mathrm{i} + a^{(7)}\mathrm{j} + a^{(8)}\mathrm{k}) \ . \end{aligned}$$

Addition of dual quaternions is usually defined as component-wise addition. Multiplication is defined as

$$\begin{aligned} \underline{a} \boxplus \underline{b} &= \begin{pmatrix} \underline{a}_q \\ \underline{a}_d \end{pmatrix} \boxplus \begin{pmatrix} \underline{b}_q \\ \underline{b}_d \end{pmatrix} \\ &= \begin{pmatrix} \underline{a}_q \oplus \underline{b}_q \\ \underline{a}_q \oplus \underline{b}_d + \underline{a}_d \oplus \underline{b}_q \end{pmatrix} \ . \end{aligned}$$

Here $\oplus$ denotes the well-known quaternion multiplication. From this definition, it is easily seen that both, quaternions and dual numbers can be thought of as a special case of dual quaternions.

Consider a dual quaternion $\underline{a} = (\underline{a}_p^\top, \underline{a}_d^\top)^\top$, where $\underline{a}_p$ is the quaternion representing the non-dual part and $\underline{a}_d$ is the quaternion representing the dual part. Then, its inverse $\underline{a}^{-1}$ is given by

$$\underline{a}^{-1} = \underline{a}_p^{-1} \boxplus (1 - \varepsilon \boxplus \underline{a}_d \boxplus \underline{a}_p^{-1}) \ ,$$

where $\underline{a}_p^{-1}$ denotes the quaternion inverse of $\underline{a}_p$. From this, it can be seen that the inverse does not always exist. A necessary and sufficient condition for its existence is $\underline{a}_p \neq 0$.

This formalism is sufficient for defining representations of rigid-body motions using dual quaternions. First, a pure rotation is represented using unit quaternions in the usual way. That is, a rotation with angle θ around axis $\underline{x} = (x^{(1)}, x^{(2)}, x^{(3)})$ is (assuming $||x|| = 1$) represented by the quaternion

$$\cos\left(\frac{\theta}{2}\right) + \left(x^{(1)}\,\mathrm{i} + x^{(2)}\,\mathrm{j} + x^{(3)}\,\mathrm{k}\right) \cdot \sin\left(\frac{\theta}{2}\right) \ .$$

The dual part becomes important as soon as translations come into play. A pure translation $\underline{t} = (t_x, t_y, t_z)$ is represented by dual quaternion

$$1 + \varepsilon\,(t_x\,\mathrm{i} + t_y\,\mathrm{j} + t_z \mathrm{k}) \ .$$

Combinations of rotations and translations are obtained using dual quaternion multiplication. Using a suitable adaption of the concept of a norm, it can be shown that all dual quaternions involved in this representation have unit length. Furthermore, the dual quaternions $\underline{a}$ and $-\underline{a}$ represent the same rigid-body motion. Besides that, the representation is unique. Thus, unit dual quaternions are a double cover of the SE(3).

The restriction to the planar case can be thought of as a combination of rotations around the z-axes and translations in the x-y-plane. That is, the translation is represented by the dual quaternion

$$1 + \varepsilon \frac{1}{2}(t_x \mathrm{i} + t_y + \mathrm{j})$$

and the rotation is represented by the quaternion

$$\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\mathrm{k} \ .$$

As noted above, dual quaternion multiplication is used to combine rotations and translations. A rotation with a subsequent translation is given by

$$\begin{aligned} &\left[1 + \varepsilon\frac{1}{2}(t_x \mathrm{i} + t_y \mathrm{j})\right] \cdot \left[\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\mathrm{k}\right] \\ &\quad = \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\mathrm{k} \\ &\qquad + \frac{\varepsilon}{2}\Big[\left(\cos\left(\frac{\alpha}{2}\right) t_x + \sin\left(\frac{\alpha}{2}\right) t_y\right)\mathrm{i} \\ &\qquad\quad + \left(\cos\left(\frac{\alpha}{2}\right) t_y - \sin\left(\frac{\alpha}{2}\right) t_x\right)\mathrm{j}\Big] \ . \end{aligned}$$

Now it is easily seen, that the resulting representation of planar rigid-body motions requires only four values. Thus, $4d$ vectors will be used to denote dual quaternions representing planar rigid-body motions. Furthermore, the first two vectors are required to have unit length, i.e., a rigid-body motion is represented by $\underline{a} \in \mathbb{S}^1 \times \mathbb{R}^2 \subset \mathbb{R}^4$, where $\mathbb{S}^1$ denotes the unit circle parametrized as vectors of unit length. Accordingly, our definition of the multiplication of two unit dual quaternions $\underline{a}_1$ and $\underline{a}_2$ can be simplified to

$$\begin{pmatrix} a_1^{(1)} a_2^{(1)} - a_1^{(2)} a_2^{(2)} \\ a_1^{(1)} a_2^{(2)} + a_1^{(2)} a_2^{(1)} \\ a_1^{(1)} a_2^{(3)} - a_1^{(2)} a_2^{(4)} + a_1^{(3)} a_2^{(1)} + a_1^{(4)} a_2^{(2)} \\ a_1^{(1)} a_2^{(4)} + a_1^{(2)} a_2^{(3)} - a_1^{(3)} a_2^{(2)} + a_1^{(4)} a_2^{(1)} \end{pmatrix} \ .$$

Furthermore, there exists a matrix representation of the considered subgroup of dual quaternions. For a dual quaternion $\underline{a} \in \mathbb{R}^4$, the corresponding matrix representation is given by

$$\mathbf{Q}_a = \begin{pmatrix} a^{(1)} & a^{(2)} & 0 & 0 \\ -a^{(2)} & a^{(1)} & 0 & 0 \\ -a^{(3)} & a^{(4)} & a^{(1)} & -a^{(2)} \\ -a^{(4)} & -a^{(3)} & a^{(2)} & a^{(1)} \end{pmatrix} \ .$$

B. Uncertain Planar Rigid-Body Motions

For representing uncertainty over the considered subgroup of unit dual-quaternions, it is necessary to use a probability distribution that accounts for the fact that $\underline{a}$ and $-\underline{a}$ represent the same rigid-body motion. Thus, we propose to use a distribution that is characterized by the p.d.f.

$$f(\underline{x}) = \frac{1}{N(\mathbf{C})} \exp\left(\underline{x}^\top \mathbf{C}\, \underline{x}\right) \ , \quad \underline{x} \in \mathbb{S}^1 \times \mathbb{R}^2$$

This distribution naturally arises when conditioning the first two entries of a $4d$ Gaussian to unit length, and thus, it naturally satisfies antipodal symmetry. One of the challenges involved in handling this distribution is the computation of the normalization constant $N(\cdot)$ because a direct approach requires numerical integration over $\mathbb{S}^1 \times \mathbb{R}^2$. Thus, we will show how to simplify this computation.

As a first step, one can observe that the above p.d.f. can be written as

$$\begin{aligned} f(\underline{x}_s, \underline{x}_t) = \exp\big(&\underline{x}_s^\top \mathbf{T}_1 \underline{x}_s \\ &+ (\underline{x}_t - \mathbf{T}_2 \underline{x}_s)^\top \mathbf{C}_3 (\underline{x}_t - \mathbf{T}_2 \underline{x}_s)\big) \\ &\cdot N(\mathbf{C})^{-1} \ , \end{aligned}$$

where $\underline{x}_s \in \mathbb{S}^1$, $\underline{x}_t \in \mathbb{R}^2$ and $\mathbf{T}_1 = \mathbf{C}_1 - \mathbf{C}_2^\top \mathbf{C}_3^{-1} \mathbf{C}_2$, $\mathbf{T}_2 = -\mathbf{C}_3^{-1} \mathbf{C}_2$ with $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3 \in \mathbb{R}^{2\times 2}$ such that

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2^\top \\ \mathbf{C}_2 & \mathbf{C}_3 \end{pmatrix} \ .$$

From this representation it is now seen that $\mathbf{C}_1$ needs to be symmetric, $\mathbf{C}_2$ may be arbitrary, and $\mathbf{C}_3$ has to be symmetric negative definite in order to ensure that $f(\underline{x}_s, \underline{x}_t)$ is a well-defined probability density.

From this p.d.f. representation, one can also observe relationships to other distributions. Marginalizing out $\underline{x}_t$ yields a Bingham distribution density, i.e.,

$$f(\underline{x}_s) = \frac{1}{N_B(\mathbf{T}_1)} \exp(\underline{x}_s^\top \mathbf{T}_1 \underline{x}_s)$$

Furthermore, $\underline{x}_t$ given $\underline{x}_s = \underline{a}$ follows a Gaussian distribution $\mathcal{N}(\mathbf{T}_2 \underline{a},\, \mathbf{C}_3)$. These relationships can now be used for simplifying the computation of the normalization constant, which can be obtained as

$$N(\mathbf{C}) = 2\pi \sqrt{\det\left(-\frac{\mathbf{C}_3^{-1}}{2}\right)} \cdot N_B(\mathbf{T}_1) \ ,$$

where $N_B(\cdot)$ denotes the normalization constant of a Bingham distribution. This is still computationally burdensome but it can effectively be addressed with methods such as saddlepoint approximations [20], [23]. Furthermore, it is now easy to generate random samples $\underline{r}_i = (\underline{r}_{s,i}, \underline{r}_{t,i})^T$ of the proposed distribution by generating random samples $\underline{r}_{s,i}$ of a Bingham distribution (with parameter matrix $\mathbf{T}_1$) and then obtaining $\underline{r}_{t,i}$ by sampling from $\mathcal{N}(\mathbf{T}_2 \underline{r}_{s,i},\, -0.5\mathbf{C}_3)$.

A procedure for estimating the parameter matrix $\mathbf{C}$ from samples $\underline{r}_i = (\underline{r}_{s,i}, \underline{r}_{t,i})^T$ (with $i = 1, \ldots, m$) can be derived by obtaining an estimate of $\mathbf{T}_1$ first. This is carried out by applying a method for estimation of Bingham distribution parameters (e.g., numerical moment matching as is done in [7]) to determine $\underline{r}_{s,i}$. Then, multivariate linear regression can be used for obtaining $\mathbf{T}_2$ and $\mathbf{C}_3$ (see [24, Theorem 8.2.1]). The entire resulting procedure is visualized in Algorithm 1, where EstimateBingham is used to denote a procedure for estimating Bingham distribution parameters.

Algorithm 1 Parameter Estimation

procedure ESTIMATEPARAMETERS($\underline{r}_1, \ldots, \underline{r}_m$)
- $\hat{\mathbf{T}}_1 \leftarrow \text{EstimateBingham}(\underline{r}_{s,1}, \ldots, \underline{r}_{s,m})$;
- $\mathbf{\Sigma}_1 \leftarrow \sum_{i=1}^m \underline{r}_{t,i}^\top \cdot \underline{r}_{s,i}$;
- $\mathbf{\Sigma}_2 \leftarrow \sum_{i=1}^m \underline{r}_{s,i}^\top \cdot \underline{r}_{s,i}$;
- $\hat{\mathbf{T}}_2 \leftarrow \mathbf{\Sigma}_1 \mathbf{\Sigma}_2^{-1}$;
- $\hat{\mathbf{C}}_3 \leftarrow \left(\frac{-2}{N} \sum_{i=1}^m \left(\underline{r}_{t,i} - \hat{\mathbf{T}}_2 \underline{r}_{s,i}\right) \cdot \left(\underline{r}_{t,i} - \hat{\mathbf{T}}_2 \underline{r}_{s,i}\right)^\top \right)^{-1}$
- $\hat{\mathbf{C}}_2 \leftarrow -\hat{\mathbf{C}}_3\, \hat{\mathbf{T}}_2$;
- $\hat{\mathbf{C}}_1 \leftarrow \hat{\mathbf{T}}_1 + \hat{\mathbf{C}}_2^\top \hat{\mathbf{C}}_3^{-1} \hat{\mathbf{C}}_2$;
- $\hat{\mathbf{C}} \leftarrow \begin{pmatrix} \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2^\top \\ \hat{\mathbf{C}}_2 & \hat{\mathbf{C}}_3 \end{pmatrix}$;
- **return** $\hat{\mathbf{C}}$;

end procedure

Handling of weighted samples is achieved by using a suitable variant of the Bingham parameter estimator and introducing weights in the computation of $\mathbf{\Sigma}_1$, $\mathbf{\Sigma}_2$, and $\hat{\mathbf{C}}_3$.

III. DETERMINISTIC SAMPLING

Computing expectations plays a crucial role in statistical algorithms, e.g., for parameter estimation. In case of the distribution considered in the preceding section, this unfortunately requires numerical integration involving repeated evaluations of the probability density. In order to make approximate computation of $\mathbb{E}(g(\underline{x}))$ (where $\underline{x}$ is assumed to follow the distribution discussed above) feasible, we propose an easy scheme that approximates the continuous distribution defined above by a discrete distribution defined on the same domain.

Our proposed scheme is based on deterministic sampling schemes for both, the Bingham and the Gaussian distribution by making use of the relationship that was discussed in the preceding section. In case of the Gaussian, there are several deterministic schemes that have been used in the context of stochastic filtering. Some of them are listed in Table I. For the Bingham distribution, a sampling scheme reminiscent of the UKF has been proposed in [7].

Scheme	Samples	References
Gauss Filter	$L \cdot n$	[25]
S2KF	L	[26]
Spherical Simplex UKF	$n+2$	[27]
UKF	$2n+1$	[8], [28], [29]

TABLE I: Sampling schemes for approximating Gaussians of dimension n. Whenever applicable, the number of samples may be influenced by providing a user-defined parameter $L \in \mathbb{N}$.

The resulting deterministic sampling scheme is in some sense similar to the random sampling approach proposed above. First, deterministic sampling of the Bingham distribution is performed which yields samples $\underline{b}_i$ $(i = 1, \dots, n)$ with corresponding weights $\underline{w}_{b,i}$. Then, a $\mathcal{N}(\underline{0}, -0.5\mathbf{C}_3)$ distribution is deterministically sampled which yields the sample set $\underline{n}_j$ $(j = 1, \dots, m)$ with corresponding weights $\underline{w}_{n,j}$. Finally, for each sample of the Bingham distribution, we create a copy of the entire Gaussian sample set and reposition it such that its mean becomes $\mathbf{T}_2\underline{b}_i$. The weights for each resulting sample are computed as a product of the respective sample weights. This yields a set of $N \cdot M$ deterministically obtained samples. This entire procedure is given in Algorithm 2. There, SampleBingham and SampleGaussian represent procedures for deterministic sampling of the Bingham and the Gaussian distributions respectively. Furthermore, the method ExtractSubmatrices is a short form notation for extracting $\mathbf{C}_i$ from $\mathbf{C}$.

Algorithm 2 Deterministic Sampling

```
procedure DeterministicSampling(C)
    (C_1, C_2 C_3) ← ExtractSubmatrices(C)
    T_1 ← C_1 − C_2^⊤ C_3^{-1} C_2
    T_2 ← −C_3^{-1} C_2
    (b_i, w_{b,i})_{i=1,...,N} ← SampleBingham(T_1)
    (n_i, w_{n,i})_{i=1,...,M} ← SampleGaussian(0, −(1/2) C_3)
    k ← 1;
    for i ∈ {1, ..., N} do
        for j ∈ {1, ..., M} do
            s_k ← ( b_i ; n_j + T_2 · b_i );
            w_k ← w_{b,i} · w_{n,j};
            k ← k + 1;
        end for
    end for
    return (w_k, s_k)_{k=1,...,N·M};
end procedure
```

IV. Stochastic Filter for Planar Rigid-Body Motions

The filter proposed in this work assumes the considered system to follow the dynamics

$$\underline{x}_{t+1} = \underline{a}(\underline{x}_t, \underline{w}_t) \ ,$$

where $\underline{x}_t$, $\underline{w}_t \in \mathbb{S}^1 \times \mathbb{R}^2$ are dual quaternions representing the system state and the system noise. Furthermore, for better representation we assume a fairly simple noisy direct measurement model, i.e.,

$$\underline{z}_t = \underline{x}_t \boxplus \underline{v}_t \ ,$$

where the measurement $\underline{z}_t$ and the noise terms $\underline{v}_t$ are assumed to be defined on the same domain. Consideration of more complicated noise models is possible by using the methodology described in [30].

As usual, the proposed filter is subdivided into a prediction and an update step. All arising uncertainties are modeled using the distribution discussed above. That is, the current estimate is represented as a density of the considered distribution model in terms of its parameter matrix $\mathbf{C}_t^e$. The noise distributions are represented as $\mathbf{C}^w$ and $\mathbf{C}^v$, respectively (we assume time-invariant noise models for notational convenience). A point estimate $\underline{x}_t^e$ can be obtained by computing the mode $\underline{x}_{s,t}^e$ of the corresponding Bingham distribution (which is also discussed in [7]) and then computing $\mathbf{T}_2$.

A. Prediction

The prediction algorithm is subdivided into three steps. First, we perform deterministic sampling of the distribution representing the system state. Second, these samples are propagated through the system model $\underline{a}(\cdot, \cdot)$. Finally, parameter estimation is performed in order to obtain a prediction of the system state in terms of the parameter matrix $\mathbf{C}_{t+1}^p$. The entire resulting procedure is shown in Algorithm 3.

Algorithm 3 Prediction

```
procedure Predict(C_t^e, C^w)
    (s_{x,i}, p_{x,i})_{i=1,...,m} ← DeterministicSampling(C_t^e)
    (s_{w,i}, p_{w,i})_{i=1,...,m} ← DeterministicSampling(C^w)
    k ← 1;
    for i ∈ {1, ..., m} do
        for j ∈ {1, ..., m} do
            p_k ← p_{x,i} · p_{w,j};
            s_k ← a(s_{x,i}, s_{w,j});
            k ← k + 1;
        end for
    end for
    C_{t+1}^p ← EstimateParameters((s_i, p_i)_{i=1,...,m^2})
    return C_{t+1}^p;
end procedure
```

At this point, it is important to note that this entire step is approximate even for very simple system models. A useful example that provides some understanding of this problem is obtained by considering the dual quaternion product $\underline{a}_1 \boxplus \underline{a}_2$ of two random vectors $\underline{a}_1$, $\underline{a}_2$ that are distributed according to the considered distribution. It can be shown, that this product is in general a random variable that follows a probability distribution that differs from the considered model.

B. Update

For derivation of the update step, the observation model is reformulated as follows

$$\underline{x}_t^{-1} \boxplus \underline{z} = \underline{v}_t \ .$$

Then, using Bayes theorem, the updated density f_t^e is obtained as

$$f_t^e(\underline{x}) \propto f^v(\underline{x}^{-1} \boxplus \underline{z}) \cdot f_t^p(\underline{x}) \ ,$$

where f_t^p and f^v denote the densities of $\underline{x}_t^p$ and $\underline{v}_t$ respectively. Our goal is to show that f_t^e belongs to the considered distribution. Furthermore, we want to obtain a method for computing $\mathbf{C}_t^e$.

We note that an inverted unit dual quaternion $\underline{a}^{-1}$ can be reformulated [6] as $\mathbf{D}\,\underline{a}$ (with $\mathbf{D} = \mathrm{diag}(1, -1, -1, -1)$), i.e., in the considered scenario inversion corresponds to changing the sign of the last three entries. Furthermore, it can be shown that $\underline{a} \boxplus \underline{b}$ can be reformulated as $\mathbf{D}\,\mathbf{Q}_a\,\mathbf{D}\,\underline{b}$, where $\mathbf{D}$ is defined as before and $\mathbf{Q}_a$ is the matrix representation of the dual quaternion $\underline{a}$. These insights can be used for obtaining

$$f^v(\underline{x}_t^{-1} \boxplus \underline{z}) = f^v(\mathbf{Q}_{z,t}^{-1}\,\mathbf{D}\,\underline{x}_t) \ ,$$

where $\mathbf{Q}_z^{-1}$ denotes the matrix representation of the dual quaternion $\underline{z}_t^{-1}$. Finally, we obtain f_t^e as

$$\begin{aligned} f_t^e(\underline{x}) \propto & f^v(\mathbf{Q}_{z,t}^{-1}\,\mathbf{D}\,\underline{x}) \cdot f_t^p(\underline{x}) \\ \propto & \exp\left((\mathbf{Q}_{z,t}^{-1}\,\mathbf{D}\,\underline{x})^\top \mathbf{C}^v \left(\mathbf{Q}_{z,t}^{-1}\,\mathbf{D}\,\underline{x}\right)\right) \\ & \cdot \exp\left(\underline{x}^\top\,\mathbf{C}_t^p\,\underline{x}\right) \\ = & \exp\left(\underline{x}^\top\,(\mathbf{D}\,\mathbf{Q}_{z,t}^{-\top}\,\mathbf{C}^v\,\mathbf{Q}_{z,t}^{-1}\,\mathbf{D} + \mathbf{C}_t^p)\,\underline{x}\right) \ . \end{aligned}$$

From this computation, it is now seen that f_t^e belongs to the desired distribution family. The entire resulting algorithm is given in Algorithm 4.

Algorithm 4 Update

1: **procedure** MEASUREMENTUPDATE($\mathbf{C}_t^p$, $\mathbf{C}^v$, $\underline{z}$)
2: $\quad\mathbf{Z} \leftarrow$ MatrixRepresentation($\underline{z}^{-1}$);
3: $\quad\mathbf{D} \leftarrow \mathrm{diag}(1, -1, -1, -1)$;
4: $\quad\mathbf{C}_t^e \leftarrow \mathbf{D}\,\mathbf{Q}_{z,t}^{-\top}\,\mathbf{C}^v\,\mathbf{Q}_{z,t}^{-1}\,\mathbf{D} + \mathbf{C}_t^p$;
5: $\quad$**return** $\mathbf{C}_t^e$;
6: **end procedure**

V. EVALUATION

For the evaluation we assumed a very simple system model given by

$$\underline{x}_{t+1} = \underline{x}_t \boxplus \underline{w}_t$$

and noisy direct measurements

$$\underline{z}_t = \underline{x}_t \boxplus \underline{v}_t \ .$$

The entire ground truth was generated using the proposed distribution. For the initial value $\underline{x}_0$, the corresponding parameter matrix was given by $\mathbf{C}_0 = \mathrm{diag}(-10, 0, 1, 1)$. The system and measurement noise were modeled using $\mathbf{C}^w = \mathrm{diag}(0, -55, -100, -100)$ and $\mathbf{C}^v = \mathrm{diag}(0, -30, -10, -10)$ respectively. The deterministic sampling procedure used the approach from [7] (with $\lambda = 0.5$) for deterministically approximating the Bingham distribution and a naïve implementation of the UKF for approximation of Gaussians.

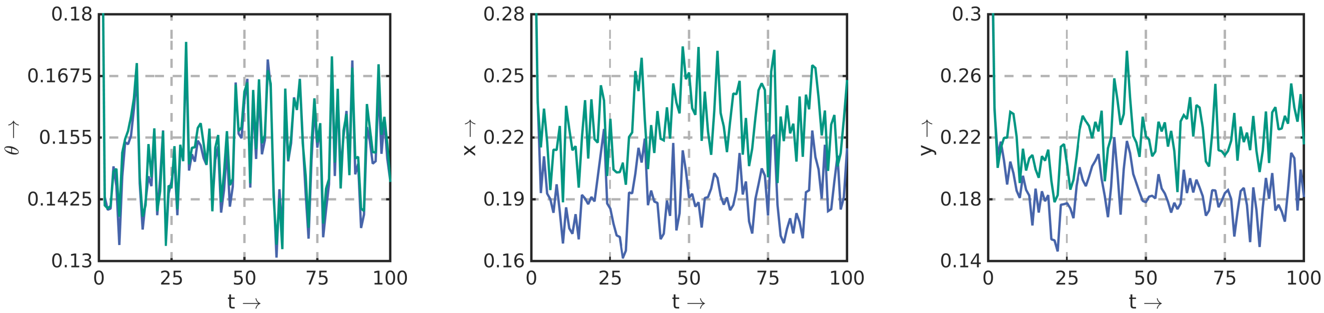
The proposed filter was compared to the UKF. Some modifications had to be introduced to the UKF in order to give it a fair chance to compete against the proposed filter. First, we helped the UKF deal with the fact that the dual quaternions $\underline{a}$ and $-\underline{a}$ represent the same orientation by introducing an intelligent repositioning of measurements. This was carried out by multiplying the obtained measurement $\underline{z}_t$ with -1 if $-\underline{z}_t$ is closer to the expected measurement $\hat{\underline{z}}_t$. Second, we projected the prediction and estimate of the UKF back to the manifold in order to obtain a feasible result. In order to obtain the parameters that were used in the UKF, random sampling was performed with a subsequent parameter estimation. This was implemented in a way that considers the repositioning procedure, i.e., we ensured the random samples to be on the same side.

We used 100 runs of the filter with 100 time steps in each run. Within these runs, we recovered the orientation and translation from each estimate. The mean error (i.e. deviation from the true system state) was used as an error measure. All results are visualized in figure 1. It is important to note that even though the superiority of the proposed filter seems quite small when it comes to orientation, the strong nonlinear relationship between position and orientation that is inherent to the dual quaternion representation has a considerable impact on the position estimate. That is, the proposed filter benefits from the sound consideration of the underlying domain.

VI. DISCUSSION AND OUTLOOK

This work presented a novel dynamic state estimator for the estimation of planar rigid-body motions. Its main difference from the state of the art is the use of a probability distribution that is inherently suitable for the representation of uncertainties on the considered domain. This not only offers the possibility of a sound representation of dependencies between position and orientation, but also avoids errors that arise due to approximation when making use of local linearity.

There are still many interesting questions for further research. First, it is of interest to gain a better understanding of the newly proposed distribution, e.g., by deriving further estimators and results on their optimality. First, for certain simple system models, parameter estimation based on moment matching might be used to avoid the need for deterministic sampling. Second, the newly proposed filter is currently restricted to the planar case. However, many applications require consideration of rigid-body motions in three-dimensional space. This presents a challenge in the present approach, as dual unit quaternions for representation of rigid-body motions in 3d might require consideration of an additional constraint in the probability distribution. Finally, it is also of considerable interest to investigate other directional distributions for this problem because the choice of the underlying probability distribution has considerable impact on the capabilities of the filter to model certain dependency relationships.



(a) Orientation (b) x-Position (c) y-Position

Fig. 1: Mean error after 100 runs in the low noise scenario showing the proposed approach (blue) and the adapted UKF (green).

References

[1] S. J. Julier and J. J. LaViola, Jr., "On Kalman Filtering With Nonlinear Equality Constraints," *IEEE Transactions on Signal Processing*, vol. 55, no. 6, Jun. 2007.

[2] N. I. Fisher, *Statistical Analysis of Circular Data*. Cambridge University Press, 1995.

[3] K. V. Mardia and P. E. Jupp, *Directional Statistics*, 1st ed. Wiley, 1999.

[4] I. Gilitschenski, "Deterministic Sampling for Nonlinear Dynamic State Estimation," PhD Thesis, Karlsruhe Institute of Technology (KIT), 2015.

[5] W. K. Clifford, "Preliminary Sketch of Biquaternions," *Proceedings of the London Mathematical Society*, vol. s1-4, no. 1, pp. 381–395, 1873.

[6] I. Gilitschenski, G. Kurz, S. J. Julier, and U. D. Hanebeck, "A New Probability Distribution for Simultaneous Representation of Uncertain Position and Orientation," in *Proceedings of the 17th International Conference on Information Fusion (Fusion 2014)*, Salamanca, Spain, Jul. 2014.

[7] ——, "Unscented Orientation Estimation Based on the Bingham Distribution (accepted)," *IEEE Transactions on Automatic Control*, Jan. 2016.

[8] S. Julier, J. Uhlmann, and H. Durrant-Whyte, "A New Approach for Filtering Nonlinear Systems," in *Proceedings of 1995 American Control Conference - ACC'95*, vol. 3. American Autom Control Council, 1995, pp. 1628–1632.

[9] X. Yun, M. Lizarraga, E. Bachmann, and R. McGhee, "An Improved Quaternion-based Kalman Filter for Real-Time Tracking of Rigid Body Orientation," in *Proceedings 2003 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS 2003)*, vol. 2. IEEE, 2003, pp. 1074–1079.

[10] A. M. Sabatini, "Quaternion-based Extended Kalman Filter for Determining Orientation by Inertial and Magnetic Sensing," *IEEE transactions on bio-medical engineering*, vol. 53, no. 7, pp. 1346–56, Jul. 2006.

[11] J. Marins, E. Bachmann, R. McGhee, and M. Zyda, "An Extended Kalman Filter for Quaternion-based Orientation Estimation Using MARG Sensors," in *Proceedings 2001 IEEE/RSJ International Conference on Intelligent Robots and Systems*, vol. 4. IEEE, 2001, pp. 2003–2011.

[12] Y. S. Suh, "Orientation Estimation Using a Quaternion-Based Indirect Kalman Filter With Adaptive Estimation of External Acceleration," *IEEE Transactions on Instrumentation and Measurement*, vol. 59, no. 12, pp. 3296–3305, Dec. 2010.

[13] E. Kraft, "A Quaternion-based Unscented Kalman Filter for Orientation Tracking," in *Proceedings of the Sixth International Conference of Information Fusion (Fusion 2003)*, Cairns, Queensland, Australia, 2003, pp. 47–54.

[14] J. S. J. Goddard, "Pose and Motion Estimation from Vision Using Dual Quaternion-based Extended Kalman Filtering," Ph.D. dissertation, University of Tennessee, Jan. 1997.

[15] A. Willsky and J. Lo, "Estimation for rotational processes with one degree of freedom–Part II: Discrete-time processes," *IEEE Transactions on Automatic Control*, vol. 20, no. 1, pp. 22–30, Feb. 1975.

[16] M. Azmani, S. Reboul, J.-B. Choquel, and M. Benjelloun, "A Recursive Fusion Filter for Angular Data," in *International Conference on Robotics and Biometrics (ROBIO)*, Dec. 2009.

[17] G. Kurz, I. Gilitschenski, and U. D. Hanebeck, "Recursive Bayesian Filtering in Circular State Spaces," *arXiv preprint: Systems and Control (cs.SY)*, Jan. 2015.

[18] J. Glover and L. P. Kaelbling, "Tracking the Spin on a Ping Pong Ball with the Quaternion Bingham Filter," in *Proceedings of the 2014 IEEE Conference on Robotics and Automation (ICRA 2014)*, Hong Kong, China, 2014.

[19] W. Feiten, P. Atwal, R. Eidenberger, and T. Grundmann, "6D Pose Uncertainty in Robotic Perception," in *Advances in Robotics Research*, T. Kröger and F. M. Wahl, Eds. Springer Berlin Heidelberg, 2009.

[20] A. Kume, S. P. Preston, and A. T. A. Wood, "Saddlepoint Approximations for the Normalizing Constant of Fisher-Bingham Distributions on Products of Spheres and Stiefel Manifolds," *Biometrika*, vol. 100, no. 4, pp. 971–984, Aug. 2013.

[21] J. B. Kuipers, *Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality*. Princeton University Press, 2002.

[22] C. Bingham, "An Antipodally Symmetric Distribution on the Sphere," *The Annals of Statistics*, vol. 2, no. 6, Nov. 1974.

[23] I. Gilitschenski, G. Kurz, S. J. Julier, and U. D. Hanebeck, "Efficient Bingham Filtering based on Saddlepoint Approximations," in *Proceedings of the 2014 IEEE International Conference on Multisensor Fusion and Information Integration (MFI 2014)*, Beijing, China, Sep. 2014.

[24] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*. Wiley-Interscience, 2003.

[25] M. F. Huber and U. D. Hanebeck, "Gaussian Filter based on Deterministic Sampling for High Quality Nonlinear Estimation," in *Proceedings of the 17th IFAC World Congress (IFAC 2008)*, vol. 17, no. 2, Seoul, Republic of Korea, Jul. 2008.

[26] J. Steinbring and U. D. Hanebeck, "S2KF: The Smart Sampling Kalman Filter," in *Proceedings of the 16th International Conference on Information Fusion (Fusion 2013)*, Istanbul, Turkey, Jul. 2013.

[27] S. Julier, "The Spherical Simplex Unscented Transformation," in *Proceedings of the 2003 American Control Conference, 2003.*, vol. 3. IEEE, 2003, pp. 2430–2434.

[28] S. Julier, J. Uhlmann, and H. Durrant-Whyte, "A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators," *IEEE Transactions on Automatic Control*, vol. 45, no. 3, pp. 477–482, Mar. 2000.

[29] S. Julier and J. Uhlmann, "Unscented Filtering and Nonlinear Estimation," *Proceedings of the IEEE*, vol. 92, no. 3, 2004.

[30] I. Gilitschenski, G. Kurz, and U. D. Hanebeck, "Non-Identity Measurement Models for Orientation Estimation Based on Directional Statistics," in *Proceedings of the 18th International Conference on Information Fusion (Fusion 2015)*, Washington D. C., USA, Jul. 2015.